

Exam demo-version

1. (10%) Evaluate the following limit:

$$\lim_{x \rightarrow 0} \sqrt[x]{\cos \sqrt{x}}$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x) = \operatorname{sgn} \left(\sin \left(\frac{\pi}{x} \right) \right).$$

3. It is known, that the 2×2 matrix A has $\operatorname{tr}(A) = -7$ (matrix trace, the sum of diagonal elements) and $\det(A) = 0$.

(a) (5%) Find the eigenvalues of A

(b) (5%) Find B^{2017} for $B = \begin{pmatrix} 1 & -4 \\ 2 & -8 \end{pmatrix}$

4. It is known that A is a square matrix and

$$A^T X A = \begin{pmatrix} 18 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 10 \end{pmatrix},$$

(a) (3%) Find $\operatorname{rank}(X)$

(b) (2%) For given matrix $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 5 & 1 \end{pmatrix}$ find the determinant $\det(X)$

(c) (5%) For given matrix $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 5 & 1 \end{pmatrix}$ find the matrix X and test it for positive and negative definiteness.

5. (10%) Solve the differential equation:

$$y''' - 4y'' + y' = 2x^2 + 1.$$

6. (10%) Find the points of maximum of the function

$$F(u, v) = \sqrt{u}(\sqrt{u} - 2) - \sqrt{v}(\sqrt{v} - 2),$$

given that $\sqrt{u} \leq 2$, $\sqrt{v} \leq 2$

7. There are three coins in the bag. Two coins are unbiased, and for the third coin the probability of «head» is equal to 0.8. James Bond chooses one coin at random from the bag and tosses it
- (5%) What is the probability that it will show «head»?
 - (5%) What is the conditional probability that the coin is unbiased if it shows «head»?
8. The pair of random variables X and Y with $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 1$ has the following covariance matrix

$$\begin{pmatrix} 10 & -2 \\ -2 & 9 \end{pmatrix}.$$

- (5%) Find $\text{Var}(X + Y)$, $\text{Corr}(X, Y)$, $\text{Cov}(X - 2Y + 1, 7 + X + Y)$
 - (5%) Find the value of a if it is known that X is independent of $Y - aX$.
9. You have height measurements of a random sample of 100 persons, y_1, \dots, y_{100} . It is known that $\sum_{i=1}^{100} y_i = 15800$ and $\sum_{i=1}^{100} y_i^2 = 2530060$.
- (3%) Calculate unbiased estimate of population mean and population variance of the height
 - (3%) At 4% significance test the null-hypothesis that the population mean is equal to 155 cm, against two-sided alternative.
 - (2%) Find the p-value
 - (2%) Find the 96% confidence interval for the population mean
10. Density function of a random variable Y is given by

$$f(y) = \begin{cases} \frac{1}{\theta^2} y e^{-y/\theta}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

You have 3 observations on Y : $y_1 = 48, y_2 = 50, y_3 = 52$.

- (4%) Using maximum likelihood, find the estimate of θ
- (3%) Is the estimator $\hat{\theta}$ unbiased?
- (3%) Calculate the variance of $\hat{\theta}$

Good luck!

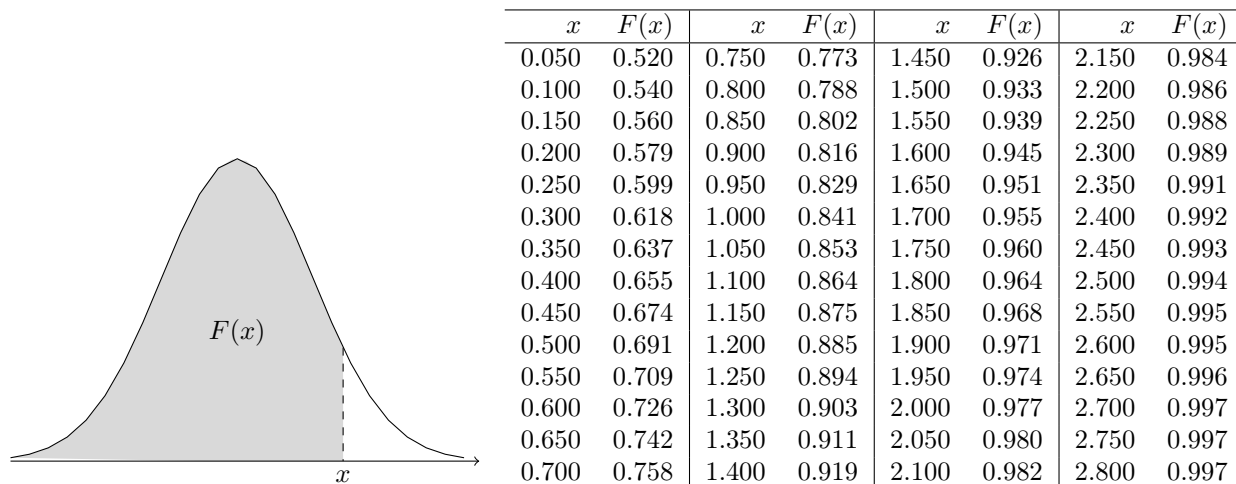


Рис. 1: Distribution function of a standard normal random variable