**Exam demo-version**

1. (10%) Evaluate the following limit:

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}}$$

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}} = \lim_{x\to 0} \left(\cos\sqrt{x}\right)^{1/x} = \lim_{x\to 0} \exp\left(\frac{\ln(\cos\sqrt{x})}{x}\right) = \exp\left(\lim_{x\to 0}\frac{\ln(\cos\sqrt{x})}{x}\right),$$

In the last equality we interchanged limit and continuous function. Now we use Taylor's expansion:

$$\frac{\ln(\cos\sqrt{x})}{x} = \frac{\ln(1-\frac{x}{2}+\frac{x^2}{24}+o(x^2))}{x} = \frac{-\frac{x}{2}+o(x)}{x}$$

It follows that

$$\lim_{x\to 0}\frac{\ln(\cos\sqrt{x})}{x} = -\frac{1}{2},$$

And finally

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}} = \exp\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{e}}.$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{x}\right)\right).$$

The discontinuity points are: the point $x = 0$, as denominator is zero, and the points $x = 1/k, k \in \mathbb{Z}$, as $\sin\left(\frac{\pi}{x}\right)$ changes sign.

At the points $x = 1/k, k \in \mathbb{Z}$ the function has first order discontinuities, as one-side limits exist but are not equal.

For example, let's consider $k = 1$. In a small right neighbourhood of the point $x = 1$, the function $\sin\left(\frac{\pi}{x}\right)$ is positive, as for $x \in (1, 2)$ one has $\frac{\pi}{2} < \frac{\pi}{x} < \pi$. For all points of this neighbourhood one has $f(x) = 1$, hence, $\lim_{x\to 1+0} f(x) = 1$.

In a small right neighbourhood of the point $x = 1$, the function $\sin\left(\frac{\pi}{x}\right)$ is negative as for $x \in (1/2, 1)$ one has $\pi < \frac{\pi}{x} < 2\pi$. For all points of this neighbourhood one has $f(x) = -1$, hence, $\lim_{x\to 1-0} f(x) = -1$. Similar neighbourhoods may be found for other values of $k$.

At the point $x = 0$ the function has second order discontinuity, as one sided limits do not exist. Let's consider the sequences $a_n = \frac{2}{1+4n}, n \in \mathbb{N}$ and $b_n = \frac{2}{3+4n}, n \in \mathbb{N}$, convergint to zero from the right side. Then

$$f(a_n) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{\frac{2}{1+4n}}\right)\right) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{2}+2\pi n\right)\right) = 1$$

And

$$f(b_n) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{\frac{2}{3+4n}}\right)\right) = \operatorname{sgn}\left(\sin\left(\frac{3\pi}{2}+2\pi n\right)\right) = -1.$$

We have shown that the limit of $f(x)$ for $x$ converging to zero from the right does not exist. One may prove that the limit from the left does not exist by considering sequences $-a_n$ and $-b_n$.

3. Matrix $A$ is given by

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

(a) (6%) Find the eigenvalues and eigenvectors of $A$;

First, we solve the equation $\det(A - \lambda I) = 0$. The roots are $\lambda_1 = 2$, $\lambda_2 = 5$, $\lambda_3 = 10$.
The eigenvectors are

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix};\ v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};\ v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

(b) (4%) Find the eigenvalues of $4A^{-1} + 2I$, where $I$ is identity matrix.

Eigenvalues of $A^{-1}$ are 0.5, 0.2 and 0.1. So the eigenvalues of $4A^{-1} + 2I$ are 4, 2.8 and 2.4.

4. The characteristic polynomial of a matrix $B$ is given by $f(\lambda) = 6\lambda - 5\lambda^2 - \lambda^3$.

(a) (6%) Find dimensions of $B$, $\operatorname{rank} B$, $\det B$, sum of diagonal elements of $B$;

The highest power of $\lambda$ is 3, so the dimension of $B$ is $3 \times 3$. We can solve for $\lambda$ and find $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -6$. So, $\det B = 0$ and $\operatorname{trace} B = \sum b_{ii} = 0 + 1 - 6 = -5$.

(b) (2%) Suppose additionaly that $B$ is symmetric. Can we find random variables with covariance matrix $B$?

The quadratic form $B$ is indefinite, so $B$ is not a valid covariance matrix.

(c) (2%) Suppose additionaly that $B$ is symmetric. How many solutions does equation $v^T B v = -2018$ has?

The quadratic form $B$ is indefinite, so the equation has infinitely many solutions.

5. (10%) Solve the following differential equations

(a) (5%) $y'' - 2y' - 8y = 0$,

Let us solve the characteristic equation

$$\lambda^2 - 2\lambda - 8 = 0$$

corresponding to the differential equation (a). It is easy to see that $\lambda_1 = 4$ and $\lambda_2 = -2$ are the solutions of this characteristic equation. Hence, the general solution of the differential equation (a) is

$$y_a(x) = C_1 e^{4x} + C_2 e^{-2x}, \quad \text{where } C_1, C_2 \in \mathbb{R}.$$

(b) (5%) $y'' - 2y' - 8y = e^x - 8\cos 2x$.

To solve the differential equation (b) we have to find the particular solutions of the following differential equations

$$y'' - 2y' - 8y = e^x, \tag{1}$$

and

$$y'' - 2y' - 8y = -8\cos 2x. \tag{2}$$

We seek the particular solution of the differential equation (1) in the form

$$y(x) = Ae^x. \tag{3}$$

Substituting expression (3) into equation (1), we obtain $A = -1/9$.
We seek the particular solution of the differential equation (2) in the form

$$y(x) = B_1 \cos 2x + B_2 \sin 2x. \tag{4}$$

Substituting expression (4) into equation (2), we obtain $B_1 = 3/5$ and $B_2 = 1/5$.
The particular solution of the differential equation (b) is a sum of particular solutions of equations (1) and (2). Thus, the particular solution of the differential equation (b) is

$$y(x) = -\frac{1}{9}e^x + \frac{3}{5}\cos 2x + \frac{1}{5}\sin 2x.$$

Therefore, the general solution of equation (b) is

$$y_b(x) = C_1 e^{4x} + C_2 e^{-2x} - \frac{1}{9}e^x + \frac{3}{5}\cos 2x + \frac{1}{5}\sin 2x, \quad \text{where } C_1,\, C_2 \in \mathbb{R}.$$

6. (10%) Find the points of maximum of the function

$$F(u,v) = \sqrt{u}\left(\sqrt{u}-2\right) - \sqrt{v}\left(\sqrt{v}-2\right),$$

given that $\sqrt{u} \le 2,\ \sqrt{v} \le 2$

1. We use the change of variables $x = \sqrt{u},\ y = \sqrt{v}$. Using algebraic manipulations we transform $G$ into: $G(x,y) = (x-1)^2 - (y-1)^2 + 3$. Now constraints have the form $x \in [0,2],\ y \in [0,2]$
2. First we check for internal extrema:

$$\frac{\partial G(x,y)}{\partial x} = 2(x-1), \ \frac{\partial G(x,y)}{\partial y} = -2(y-1), \ \frac{\partial^2 G(x,y)}{\partial x \partial y} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

   Using Sylvester's criterion we find that the point $(1,1)$ is not a maximum of $G(x,y)$.
3. Now we check for corner solutions.

   Line $\{x = 0,\ y \in [0,2]\}$, $G(0,y) = 1 - (y-1)^2$. At the point (0, 1) we have a **maximum** equal to 1, with values at the borders equal to 0.

   Line $\{y = 0,\ x \in [0,2]\}$, $G(x,0) = (x-1)^2 - 1$. At the point (1, 0) we have a minimum equal to (−1), with values at the borders equal to 0.

   Line $\{x = 2,\ y \in [0,2]\}$, $G(0,y) = 1 - (y-1)^2$. At the point (2, 1) we have a **maximum** equal to 1, with values at the borders equal to 0.

   Line $\{y = 2,\ x \in [0,2]\}$, $G(x,2) = (x-1)^2 - 1$. At the point (1, 2) we have a minimum equal to (−1), with values at the borders equal to 0.
4. Now we do inverse substitution.

Answer: Maxumum points: (0, 1) и (4, 1)

7. There are three coins in the bag. Two coins are unbiased, and for the third coin the probability of «head» is equal to 0.8. James Bond chooses one coin at random from the bag and tosses it

(a) (5%) What is the probability that it will show «head»?

$$\mathbb{P}(B) = \mathbb{P}(\text{head}) = \frac{2}{3} \cdot 0.5 + \frac{1}{3} \cdot 0.8 = \frac{18}{30} = \frac{3}{5} = 0.6$$

(b) (5%) What is the conditional probability that the coin is unbiased if it shows «head»?

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{2}{3} \cdot 0.5}{0.6} = \frac{10}{18} = \frac{5}{9}$$

8. The pair of random variables $X$ and $Y$ with $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 1$ has the following covariance matrix

$$\begin{pmatrix} 10 & -2 \\ -2 & 9 \end{pmatrix}.$$

(a) (5%) Find $\mathbb{V}\text{ar}(X+Y)$, $\mathbb{C}\text{orr}(X,Y)$, $\mathbb{C}\text{ov}(X-2Y+1, 7+X+Y)$

$$\mathbb{V}\text{ar}(X+Y) = 10 + 9 - 2 \cdot 2 = 15$$

$$\mathbb{C}\text{orr}(X,Y) = \frac{-2}{\sqrt{10 \cdot 9}}$$

$$\mathbb{C}\text{ov}(X-2Y+1, 7+X+Y) = \mathbb{V}\text{ar}(X) - 2\,\mathbb{C}\text{ov}(Y,X) + \mathbb{C}\text{ov}(X,Y) - 2\,\mathbb{V}\text{ar}(Y) = 10 - (-2) - 18 = -6$$

(b) (5%) Find the value of $a$ if it is known that $X$ is independent of $Y - aX$.

For independent variables the covariance is equal to zero: $\mathbb{C}\text{ov}(X, Y - aX) = 0$. Hence, $a = \frac{\mathbb{C}\text{ov}(X,Y)}{\mathbb{V}\text{ar}(X)} = -0.2$.

9. You have height measurements of a random sample of 100 persons, $y_1, \ldots, y_{100}$. It is known that $\sum_{i=1}^{100} y_i = 15800$ and $\sum_{i=1}^{100} y_i^2 = 2530060$.

(a) (3%) Calculate unbiased estimate of population mean and population variance of the height

Sample mean: $\bar{y} = 15800/100 = 158$.
Unbiased estimate of the variance:

$$\hat{\sigma}^2 = \frac{\sum (y_i - \bar{y})^2}{n-1} = \frac{\sum y_i^2 - n\bar{y}^2}{n-1} = 340$$

(b) (3%) At 4% significance test the null-hypothesis that the population mean is equal to 155 cm, against two-sided alternative.

Observed value of $Z$-statistics

$$Z_{obs} = \frac{158 - 155}{\sqrt{340}/\sqrt{100}} = 1.63$$

Critical value of $Z$-statistics $Z_{crit} = 2.05$.
Conclusion: hypothesis $H_0$ is not rejected.

(c) (2%) Find the p-value

Using tables we find that the area under the curve to the right of 1.63 is approximately 5%. Hense p-value is equal to 10%.

(d) (2%) Find the 96% confidence interval for the population mean

The confidence interval has the form

$$[158 - 2.05 \cdot \sqrt{340/100}; 158 + 2.05 \cdot \sqrt{340/100}]$$

Finally: $[154.2; 161.8]$

10. Let $X = (X_1, \ldots, X_n)$ be a random sample from normal distribution with zero mean and unknown variance $\sigma^2 > 0$.

If $\xi \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[\xi^4] = 3\sigma^2$.

(a) (2%) Derive the log-likelihood function of a random sample $X$.

The likelihood function of a random sample $X$ is

$$L(x_1, \ldots, x_n; \sigma^2) = f_{X_1, \ldots, X_n}(x_1, \ldots, x_n; \sigma^2) =$$

$$= \prod_{i=1}^{n} f_{X_i}(x_i; \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} = (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}}.$$

Hence, the log-likelihood function of a random sample $X$ is

$$l(x_1, \ldots, x_n; \sigma^2) := \ln \mathcal{L}(x_1, \ldots, x_n; \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{\sum_{i=1}^n x_i^2}{2\sigma^2}.$$

(b) (2%) Find the estimator of the parameter $\sigma^2$ using maximum likelihood method.

Let us write the likelihood equation:

$$\frac{\partial l(x_1, \ldots, x_n; \sigma^2)}{\partial(\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^n x_i^2}{2\sigma^4} = 0.$$

Solving this equation with respect to $\sigma^2$, we find $\sigma^2 = \sum_{i=1}^n x_i^2$. Hence, the maximum likelihood estimator of the parameter $\sigma^2$ is $\widehat{\sigma^2}_{ML} = \frac{1}{n}\sum_{i=1}^n X_i^2$.

(c) (2%) Using the realization of a random sample $x = (1, -2, 0, 1)$ find the maximum likelihood estimate of the parameter $\sigma^2$ derived in (b).

The maximum likelihood estimate of the parameter $\sigma^2$ is

$$\widehat{\sigma^2}_{ML} = \tfrac{1}{4}\left(1^2 + (-2)^2 + 0^2 + 1^2\right) = \tfrac{3}{2}.$$

(d) (2%) Find the Fisher information $I_n(\sigma^2)$ about the parameter $\sigma^2$ contained in $n$ observations of a random sample.

Considering that $l(x_1; \sigma^2) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{x_1^2}{2\sigma^2}$, we arrive at

$$\frac{\partial l(x_1; \sigma^2)}{\partial(\sigma^2)} = -\frac{1}{2\sigma^2} + \frac{x_1^2}{2\sigma^4} = \frac{x_1^2 - \sigma^2}{2\sigma^4}.$$

Therefore, the Fisher information $I_1(\sigma^2)$ about the parameter $\sigma^2$ contained in a single observation of a random sample is

$$I_1(\sigma^2) = \mathbb{E}\left[\left(\frac{\partial l(X_1;\sigma^2)}{\partial(\sigma^2)}\right)^2\right] = \mathbb{E}\left[\frac{\left(X_1^2-\sigma^2\right)^2}{4\sigma^8}\right] = \frac{\mathbb{E}\left[\left(X_1^2-\sigma^2\right)^2\right]}{4\sigma^8} \overset{\mathbb{E}[X_1^2]=\sigma^2}{=}$$

$$= \frac{\mathrm{D}(X_1^2)}{4\sigma^8} = \frac{\mathbb{E}[X_1^4]-\left(\mathbb{E}[X_1^2]\right)^2}{4\sigma^8} = \frac{3\sigma^4-\left(\sigma^2\right)^2}{4\sigma^8} = \frac{2\sigma^4}{4\sigma^8} = \frac{1}{2\sigma^4}.$$

Thus, $I_n(\sigma^2) = n\cdot I_1(\sigma^2) = \frac{n}{2\sigma^4}$.

(e) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ an unbiased estimator of the parameter $\sigma^2$?

The estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ is unbiased, since

$$\mathbb{E}[\widehat{\sigma^2}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i^2\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i^2] = \frac{1}{n}\sum_{i=1}^n \sigma^2 = \frac{1}{n}n\sigma^2 = \sigma^2.$$

(f) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ an efficient estimator of the parameter $\sigma^2$?

The estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ is efficient, as

$$\mathrm{D}(\widehat{\sigma^2}) = \mathrm{D}\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{D}(X_i^2) = \frac{1}{n^2}\sum_{i=1}^n 2\sigma^4 = \frac{1}{n^2}n2\sigma^4 = \frac{2\sigma^4}{n},$$

and

$$I_n^{-1}(\sigma^2) = \left(\frac{n}{2\sigma^2}\right)^{-1} = \frac{2\sigma^2}{n} = \mathrm{D}(\widehat{\sigma^2}).$$

(g) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ a consistent estimator of the parameter $\sigma^2$?

As random variables $X_1^2,\ldots,X_n^2,\ldots$ are independent, have the same distribution, and have finite means, then the we can apply the law of large numbers, according to which we have

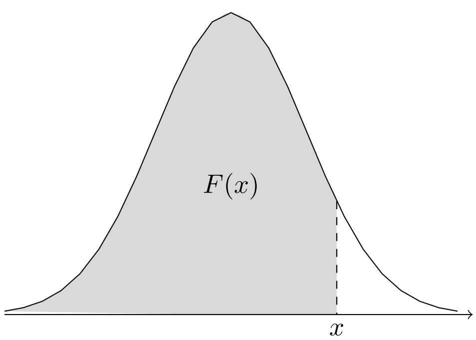
$$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_i^2] = \sigma^2 \qquad \text{as } n\to\infty.$$

(h) (2%) Find the following probability limit $\operatorname{plim}_{n\to\infty}\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$.

As $\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_i^2] = \sigma^2$, and the function $g(x)=\sqrt{x}$ is continuous, by Slutsky's theorem we derive

$$\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2} = g\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) \xrightarrow{\mathbb{P}} g(\sigma^2) = \sqrt{\sigma^2} = \sigma \qquad \text{as } n\to\infty.$$

Good luck!




| $x$ | $F(x)$ | $x$ | $F(x)$ | $x$ | $F(x)$ | $x$ | $F(x)$ |
|---|---|---|---|---|---|---|---|
| 0.050 | 0.520 | 0.750 | 0.773 | 1.450 | 0.926 | 2.150 | 0.984 |
| 0.100 | 0.540 | 0.800 | 0.788 | 1.500 | 0.933 | 2.200 | 0.986 |
| 0.150 | 0.560 | 0.850 | 0.802 | 1.550 | 0.939 | 2.250 | 0.988 |
| 0.200 | 0.579 | 0.900 | 0.816 | 1.600 | 0.945 | 2.300 | 0.989 |
| 0.250 | 0.599 | 0.950 | 0.829 | 1.650 | 0.951 | 2.350 | 0.991 |
| 0.300 | 0.618 | 1.000 | 0.841 | 1.700 | 0.955 | 2.400 | 0.992 |
| 0.350 | 0.637 | 1.050 | 0.853 | 1.750 | 0.960 | 2.450 | 0.993 |
| 0.400 | 0.655 | 1.100 | 0.864 | 1.800 | 0.964 | 2.500 | 0.994 |
| 0.450 | 0.674 | 1.150 | 0.875 | 1.850 | 0.968 | 2.550 | 0.995 |
| 0.500 | 0.691 | 1.200 | 0.885 | 1.900 | 0.971 | 2.600 | 0.995 |
| 0.550 | 0.709 | 1.250 | 0.894 | 1.950 | 0.974 | 2.650 | 0.996 |
| 0.600 | 0.726 | 1.300 | 0.903 | 2.000 | 0.977 | 2.700 | 0.997 |
| 0.650 | 0.742 | 1.350 | 0.911 | 2.050 | 0.980 | 2.750 | 0.997 |
| 0.700 | 0.758 | 1.400 | 0.919 | 2.100 | 0.982 | 2.800 | 0.997 |

Рис. 1: Distribution function of a standard normal random variable