Олимпиада НИУ ВШЭ для студентов и выпускников – 2018 г.

The time to complete the job – **180** min.

CODE - 120

Please note: You can write the answer in **English or in Russian.**

1. (10%) For the function

$$f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

If possible find the following limits:

(a) (2%) $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$;

(b) (2%) $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)$;

(c) (6%) $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$.

2. (10%) Find the discontinuity points of the following function

$$f(x)=\frac{\frac{1}{x}-\frac{1}{x+1}}{\frac{1}{x-1}-\frac{1}{x}}$$

and find the limits of $f(x)$ as x tends to these discontinuity points.

3. Matrix A is given by

$$A=\begin{pmatrix}2&1&3&4\\0&2&1&3\\2&1&6&5\\1&2&4&8\end{pmatrix}.$$

(a) (6%) Find the space V of all eigenvectors of A corresponding to the eigenvalue $\lambda=1$;

(b) (4%) Find one vector that is orthogonal to the space V;

4. The 3×3 matrix A has eigenvalues 0, 1 and 2. If there is enough information find

(a) (3%) The determinant A^TA;

(b) (4%) The eigenvalues of A^TA;

(c) (3%) The eigenvalues of $(A^3-2I)^{-1}$, where I is the identity matrix;

5. (10%) Find the point and the value of conditional maximum of the function $F(x,\ y)=\max(2x+3y,3x+2y)$ subject to $x^2+y^2=1$.

6. (a) (4%) Find the general solution of $y''-2y'+10y=0$,

(b) (2%) Find any particular solution of $y''-2y'+10y=\sin 3x$,

(c) (2%) Find any particular solution of $y''-2y'+10y=e^x$,

(d) (2%) Find the general solution of $y''-2y'+10y=\sin 3x+e^x$.

7. Time before the first screen break of IPhone 11 is a random variable, X, with exponential distribution: $\mathbb{P}(X \leq x) = 1 - e^{-\lambda x}$. The mean time before break is 10 months.
   (a) (3%) Find the probability that a new IPhone screen will not be broken during the first 15 months.
   (b) (3%) Find the variance of the time before the first screen break.
   (c) (4%) You have an IPhone with screen which was not broken during the first 10 months since purchase. Find the probability that the screen for this IPhone will be ok for at least additional 15 months.

8. Joint probability density function of random variables X and Y is:

$$f(x,y) = \begin{cases} c(x^2+y), \text{ if } 0 \leq x \leq 2, 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

   (a) (2%) Find c
   (b) (3%) Check whether X and Y are independent
   (c) (2%) Find $\mathbb{E}(X)$
   (d) (3%) Find probability $\mathbb{P}(XY > 1)$

9. Let $X = (X_1, \ldots, X_n)$ be a random sample from the distribution with density function

$$f(x, \theta) = \begin{cases} \frac{2x}{\theta^2}, & x \in [0; \theta], \\ 0, & x \notin [0; \theta], \end{cases}$$

   where $\theta > 0$ is an unknown parameter.
   (a) (2%) Find the estimator of the parameter θ using method of moments. Use first initial moment condition.
   (b) (2%) Is the estimator from (a) an unbiased estimator of the parameter θ?
   (c) (2%) Is the estimator from (a) a consistent estimator of the parameter θ?
   (d) (2%) Find the following probability limit $\text{plim}_{n\to\infty} e^{\overline{X}}$.
   (e) (2%) Find the estimator of the parameter θ using maximum likelihood method.

10. $X_1, \ldots, X_n$ is independent identically distributed sample from Bernoulli distribution with probability θ. Additionally, it is known, that $1/2 \leq \theta \leq 1$.
   (a) (2%) Find the method of moments estimator of the parameter θ.
   (b) (2%) Compute mean squared error of $\hat{\theta}_{MM}$
   (c) (4%) Find maximum likelihood estimator of parameter θ
   (d) (2%) Compute mean squared error of $\hat{\theta}_{ML}$ at $\theta = 1$

Good luck!