Профиль: "Математические методы анализа экономики" **КОД - 120**

Время выполнения - 180 мин., ответ - русский или английский.

1. (10%) Find the following limit or prove it doesn't exist

$$\lim_{x,y\to 0} \frac{1-\cos(x+2y)}{\sin(xy)}.$$

2. (10%) For the following sequence

$$x_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$$

find

$$\sup x_n, \inf x_n, \limsup_{n\to\infty} x_n, \liminf_{n\to\infty} x_n, \lim_{n\to\infty} x_n.$$

3. (10%) Peter wants to congratulate his mother and grandmother on the International Women's Day. He decided to write congratulations on homemade cards. The borders of homemade cards are curves that satisfy the equations: The card for mother

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 100$$

The card for grandmother

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & 2.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 100$$

All values given in centimeters.

What size envelopes should Peter buy to put these cards in? Suggest minimal possible size for each envelope.

4. (10%) Find the orthogonal projection of the vector $f = (1, 2, 3, 4, 5)$ on the linear subspace $L \subset \mathbb{R}^5$ generated by vectors $g_1 = (1, 1, 1, 1, 1)$, $g_2 = (0, 0, 0, 1, 1)$ and $g_3 = (0, 0, 0, 0, 1)$. Find the normal vector.

5. (10%) Find conditional extrema of the function

$$F(x_1, x_2, x_3) = 0.5x_1^2 + 5x_1x_2 + 0.5x_2^2 + x_3^2$$

subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

6. (10%) Solve the following system of differential equations:

$$\begin{cases} \ddot{x} = 2x - 3y, \\ \ddot{y} = x - 2y. \end{cases} \tag{1}$$

Answer must be represented in real form.

7. Consider some statistical test with 1% probability of Type I error and power of 60%. Researcher assumes that a priori (before the test) probability of the null hypothesis is 0.5.
   (a) (8%) If the test indicates a rejection of the null hypothesis, what is the probability that the null is false?
   (b) (2%) Briefly explain how in general probabilities of Type I and Type II errors are connected?

8. A magic shield protects Earth from alien invasion. The shield only works when it is powered by a power generator. Once one generator breaks down, another generator is turned on, and there is no way to repair the broken generator. Once the last generator breaks down, aliens start the invasion.

   As of today, there are two generators on Earth: one is currently powering the shield, the second is kept in reserve. It is known that working times of generators are independent and have exponential distributions with means 5 and 10 years for first and second generator respectively.
   (a) (1%) Find the probability that the first generator will last for at least 2 years more if it is known, that it was turned on 5 years ago.
   (b) (5%) Let Z be the time the Earth shield will last (time until the last generator breaks down). Find the probability density function of Z
   (c) (4%) The magic shield of Mars works in another fashion: it requires BOTH generators to work simultaneously. If at least one generator breaks down, the shield disappears. Find the distribution of working time of Mars shield with the same two generators as were used on Earth.

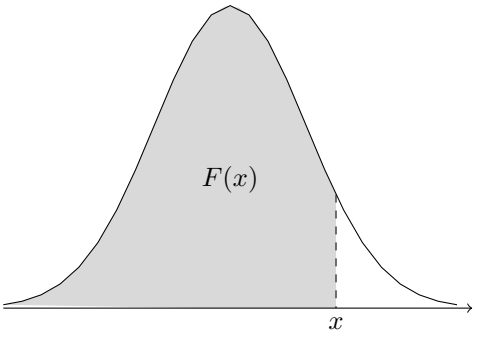
9. Anna and Bella found a biased coin, which lands on heads with unknown probability p. On the first day they flipped the coin 100 times. On the second day — 200 times. On the third day — 400 times.

   Anna noted the total number of heads for the first and second day: 120 times. Bella noted the total number of heads for the second and the third day: 300 times.
   (a) (6%) Please help Anna and Bella to construct an unbiased estimate of p with minimal variance.
   (b) (2%) Estimate the variance of the estimator you have constructed.
   (c) (2%) Find the 95% confidence interval for the unknown p.

10. Let X_1, X_2, ..., X_{100} be a random sample from normal distribution $\mathcal{N}(0; \sigma^2)$. The parameter σ^2 is unknown. We observe that $\sum_{i=1}^{100} y_i^2 = 200$.
    (a) (4%) Write the log-likelihood function and find the maximum likelihood estimate of σ^2.
    (b) (4%) Find the Fisher information and estimate it.
    (c) (2%) Find approximate 95% confidence interval for σ^2.

Good luck!

Рис. 1: Distribution function of a standard normal random variable



x	$F(x)$	x	$F(x)$	x	$F(x)$	x	$F(x)$
0.050	0.520	0.750	0.773	1.450	0.926	2.150	0.984
0.100	0.540	0.800	0.788	1.500	0.933	2.200	0.986
0.150	0.560	0.850	0.802	1.550	0.939	2.250	0.988
0.200	0.579	0.900	0.816	1.600	0.945	2.300	0.989
0.250	0.599	0.950	0.829	1.650	0.951	2.350	0.991
0.300	0.618	1.000	0.841	1.700	0.955	2.400	0.992
0.350	0.637	1.050	0.853	1.750	0.960	2.450	0.993
0.400	0.655	1.100	0.864	1.800	0.964	2.500	0.994
0.450	0.674	1.150	0.875	1.850	0.968	2.550	0.995
0.500	0.691	1.200	0.885	1.900	0.971	2.600	0.995
0.550	0.709	1.250	0.894	1.950	0.974	2.650	0.996
0.600	0.726	1.300	0.903	2.000	0.977	2.700	0.997
0.650	0.742	1.350	0.911	2.050	0.980	2.750	0.997
0.700	0.758	1.400	0.919	2.100	0.982	2.800	0.997