Профиль: "Математические методы анализа экономики" **КОД - 120**

Время выполнения - 180 мин., язык русский или английский.
Максимальное количество баллов - 100.

1. (10%) Find the following limit:
$$\lim_{x\to 0} \sin(x)\sin\left(\frac{1}{x^2}\right).$$

2. (10%) Consider the function
$$g(b) = f(f(f(b^2 f(b)))).$$
Find $g'(1)$ if $f(1) = 2, f(2) = 3, f(3) = 1, f'(1) = 3, f'(2) = 1, f'(3) = 2$.

3. (10%) The entrepreneur plans to open several points of sale and a Central warehouse for their supply on one side of the highway, which goes strictly in a straight line. For reasons of economy, he wants the shortest distance from the Central warehouse to any of the points of sale to be half as short as the shortest distance from that point to the highway. Local conditions allowed placing the Central warehouse in such a way that the shortest distance from it to the highway is 500 meters.

   1. What form will have a track connecting all possible points of location of points of sales?
   2. Whether it is possible to estimate the maximum and minimum removal from the highway of possible points of placement of points of sale and, if it is possible, what these removals will be equal.

   Instruction. It is necessary to give all the formulas for calculations and carry out arithmetic calculations. The numeric answer must be given with one character after the decimal point.

4. (10%) Let
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad H := A(A^TA)^{-1}A^T.$$
   Find

   1. (1 point) A^Tb;
   2. (1 point) A^TA;
   3. (1 point) $\det(A^TA)$;
   4. (1 point) $(A^TA)^{-1}$;
   5. (1 point) $(A^TA)^{-1}A^Tb$;
   6. (1 point) H;
   7. (2 points) eigenvalues of H;
   8. (2 points) some basis from the eigenvectors of the matrix H.

5. (10%) Find the point of the conditional minimum of the function and the minimum value itself for different values of the parameter $a > 2$.

   Function: $F(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 3)^2$.

   Condition:
$$\begin{cases} 0 \le x_2 \le 4, \\ x_2 \le ax_1, \\ x_2 \ge 2x_1, \\ x_1 \ge 0. \end{cases}.$$

6. Find a partial solution of differential equation
$$y^{(4)} + y^{(2)} = 2\cos x,$$
   that satisfies the following conditions $y(0) = -2$, $y'(0) = 1$ and $y''(0) = y'''(0) = 0$.

7. Consider the process $\varepsilon_t \sim iidN(0,1), t \in Z$. The process X_t is defined in the following manner:

$$X_t = \begin{cases} \varepsilon_t, \text{if t is odd,} \\ \frac{\sqrt{2}}{2}(\varepsilon_{t-1}^2 - 1), \text{if t is even.} \end{cases}$$

   Check whether mathematical expectation, variance and autocovariance of X_t do not depend on t. Autocovariance is a covariance between different variables of the same process, it is formally defined as $cov(X_t, X_{t-s}), s \geq 1$.
   HINT. $E(\varepsilon_t^4) = 3$.

8. Joint probability density function of random variables X and Y is:

$$f(x,y) = \begin{cases} c(x^2 + y), \text{ if } 0 \leq x \leq 2, 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

   (a) (2%) Find c
   (b) (3%) Check whether X and Y are independent
   (c) (2%) Find $\mathbb{E}(X)$
   (d) (3%) Find probability $\mathbb{P}(XY > 1)$

9. Consider a sample of independent random variables $X_1, \ldots, X_n$ with constant variance $\sigma^2 > 0$ and increasing mean $\mathbb{E}[X_i] = \beta i$. Two estimators are suggested for β.

$$\hat{\beta} = \frac{X_n - X_1}{n-1}.$$

$$\tilde{\beta} = \frac{(X_{n-1} + X_n) - (X_1 + X_2)}{2(n-2)}.$$

   (a) (5%) Is $\hat{\beta}$ consistent?
   (b) (5%) Compare the mean squared error of the two estimators for the case $n = 4$.

10. Let $X_1, \ldots, X_n$ be independent normal random variables with mean μ and variance σ^2.
    Consider two tests for $H_0 : \mu = 1, \sigma^2 = 1$.
    Test 1. Reject H_0 if $|\bar{X} - 1| > \frac{2}{3}$, where $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$.
    Test 2. Reject H_0 if $\sum_{i=1}^{9}(X_i - 1)^2 > 14.68$.
    Calculate the probability of type I error for

    (a) (5%) test 1;
    (b) (5%) test 2.

Good luck!