NATIONAL RESEARCH UNIVERSITY
"HIGHER SCHOOL OF ECONOMICS"

HSE Global Scholarship Competition 2021

Criteria, topics, preparation materials in Mathematics for students of 10^{th}-11^{th} grade

Moscow 2020

HSE Global Scholarship Competition (GSC) in Mathematics will be held both in in-person and online formats. Each participant may choose just one of the formats: writing either on a special venue or remotely.

Eather way a participant has 180 minutes for writing the test.

A task of GSC'21 in Mathematics consists of **10 problems** divided into three blocks:

- The *first block* consists of problems from 1^{st} to 6^{th}. The problems will be graded **7 points** for correct answer and 0 **points** for incorrect one.
- The *second block* problems (7^{th} and 8^{th}) will be marked in a range **from 0 to 13 points**. The answer and a proof scheme would be considered for marking.
- The *third block* problems (9^{th} and 10^{th}) will be marked in a range **from 0 to 16 points**. Entire solution is expected in this part.

You can see that problems 7 to 10 suppose writing partial or full solution. **Proof scheme** in the second block means thesis of a solution – writing key steps of the solution and avoiding too many details. Coherent text is not needed here. An example of a proof scheme you can find in the end of the file.

Those who write the competition remotely will be able to send their solution either through a special web-form or in a file of a suggested format. The *preferred* format is **.pdf**. Most of the text editors have an option to convert file into this format. Additional available formats (if any) would be specified on start of the competition.

Task complexity is mostly a subjective concept, it depends of interests and skill set of each particular competitor. Nevertheless, we can speak of averaged complexity. On average the complexity of problems will be growing from the first one to the last one. Problems of the first block are closer to school examinations and test competitor's skills in concepts and methods of school program. Problems of the second block are out of the standard school program but could be found on classes of advanced math as more technical tasks. Problems of the third block are of a high olympiad level comparable to the finals of All-Russian mathematical olympiad of International mathematical olympiad (IMO).

For solving most of the competition tasks it is enough to know school program which we mention further. However, olympiad problems oftentimes need not just knowledge but also creativity, reasoning skills, analyzing the computational results, conjecturing statements and verifying the conjectures.

Olympiads with nonstandard problems are important, more then 50-year tradition of Russian mathematical education. Many taskbooks with olympiad problems were issued during this time. Anyone can get acquainted with such problems, test himself and train from that books. Mathematicians gained much experience in solving nonstandard problems. It led to creation of an informal “olympiad minimum” program – a set (not fixed) of tricks, methods and theorems which proved to be useful on an olympiad and which are oftentimes out of school program but still could be proved with just school methods.

The main source of such “olympiad minimum” are textbooks with thematic olympiad problems collections and materials of various school mathematical circles (optional classes). Please take into account that oftentimes such textbooks show adapted to school level and polished solutions of very hard problems. Even though such reading is extremely useful for developing mathematical skills, mastering this material to this extent is not necessary for solving olympiad problems.

Basic bibliography

Fomin D., Genkin S., Itenberg I. Mathematical Circles (Russian Experience). AMS, 1996

Vilenkin N.Ya. Combinatorics. Academic Press, New York, 1971.

Prasolov V. Problems in plane and solid geometry. Plane Geometry

Additional bibliography

Fedorov R., Belov A., Kovaldzhi A., Yashchenko I.
Moscow Mathematical Olympiads, 1993-1999. AMS, 2011.

Fedorov R., Belov A., Kovaldzhi A., Yashchenko I.
Moscow Mathematical Olympiads, 2000-2005. AMS, 2011.

Taylor P. J. International Mathematics Tournament of the Towns,

Book 1: 1980-1984. Australian Mathematics Trust, 1993.
Book 2: 1984-1989. Australian Mathematics Trust, 2003.
Book 3: 1989-1993. Australian Mathematics Trust, 1994.

Taylor P. J., Storozhev A. M. International Mathematics Tournament of Towns, Book 4: 1993-1997. Australian Mathematics Trust, 2003.

Storozhev A. M. International Mathematics Tournament of Towns, Book 5: 1997-2002. Australian Mathematics Trust, 2006.

Andreescu T., Feng Z. 102 Combinatorial Problems. Birkhauser, 2002.

Andreescu T., Andrica D. Number Theory: Structures, Examples, and Problems. Birkhauser, 2009.

Andreescu T., Andrica D., Feng Z. 104 Number Theory Problems. Birkhauser, 2007.

Andreescu T., Enescu B. Mathematical Olympiad Treasures. Birkhauser, 2012.

Coxeter H. S. M., Greitzer S. L. Geometry Revisited. AMS, 1967.

Engel A. Problem-Solving Strategies. Springer, 1999.

Key notions of school program for competition in the 10th and 11th grades (as a reference for the first block of a task)

Numbers: from integers to reals, notations, axiomatic approach, divisibility of integer numbers, GCF and LCM, prime decomposition.

Functions: linear, rational, defined by conditions, exponential, trigonometric, polynomials and n-th roots, inverse function, division of polynomials with a remainder.

Function's properties: graph of a function, shifts of a graph, increasing/decreasing, odd/even functions, minimum/maximum, periodic functions.

Inequalities: linear and square, rational inequalities and the method of intervals, using properties of functions for comparing values.

Equalities: linear, square, Vieta theorem, rational equalities, Уравнения: линейные, квадратные, теорема Виета, рациональные, compound interest (compound percentage), systems of equations, trigonometric, roots of polynomials.

Sequences: arithmetic and geometric progressions, sum of elements of a progression, sequence limit.

Derivative: function limit, differentiation rules, using derivative for analysing a function, extremum points, graph tangent equation.

Complex numbers: arithmetic operations, geometric interpretation, exponentiation and taking a root.

Combinatorics and probability theory: exhaustive search, the complementary set, inclusion-exclusion principle, choose numbers/permutations/arrangements, mathematical expectation, combinatorial definition of probability, geometric probability.

Geometry of a triangle: congruence and similarity conditions, auxiliary objects inside a triangle: median, height, angle bisector, incircle and circumcircle, 'wonderful' triangle points.

Arithmetic of angles: angles and parallel lines, inscribed and central angles of a circle, angle between tangent and chord, specific marks on drawings for counting angles.

Computations in a triangle: triangle inequality, sin and cos theorems, Ceva and Menelaus theorems, expressions for basic elements of a triangle using sides and angles, area of a triangle and of more complex figures.

Vector-coordinate approach: basic vector operations, coordinate definitions for basic geometric objects, scalar product.

Geometric transformations: plane movements, movements composition, homothety.

Stereometry: sections, basic bodies and their volume, regular polyhedra, 3d vectors and coordinate approach, parallelism and perpendicularity in 3d space.

An example of a proof scheme:

Problem 7 (demonstration task HSE GSC 2016):

Find all such pairs of relatively prime integer positive a and b, that a number $2a^2 + 3b^2$ is divisible by $2a + 3b$.

Proof scheme:

1. Consider differences $2a^2 + 3b^2 - a(2a + 3b)$ and $2a^2 + 3b^2 - b(2a + 3b)$, each divisible by $2a + 3b$. Conclude $(b - a) \cdot GCF(2a, 3b)$ is divisible by $2a + 3b$.
2. Convert to the form: $(b - a) \cdot GCF(2, b) \cdot GCF(3, a)$ is divisible by $2a + 3b$.
3. If $a = b$ then as they are relatively prime, we deduce $a = b = 1$.
4. If $GCF(2, b) \cdot GCF(3, a) \leq 2$ then we get a contradiction using the following property: if a positive integer n is divisible by m then $n \geq m$.
5. If $GCF(2, b) \cdot GCF(3, a) = 3$ then by subtracting $2a + 3b$ from $(b - a) \cdot GCF(2, b) \cdot GCF(3, a)$ we get $5a$ is divisible by $2a + 3b$. It gives an extraneous solution.
6. If $GCF(2, b) \cdot GCF(3, a) = 6$ then similarly we get $10a$ is divisible by $2a + 3b$.
7. Using substitution $b = 2b'$ and $a = 3a'$ we get 5 is divisible by $a' + b'$. Direct check gives two extraneous solutions and two proper solutions: (3, 8) and (9, 4).

Answer: the solutions are (1,1); (3,8); (9,4).