HSE Global Scholarship Competition – 2021
Mathematics
10th grade, variant 1

1. The recommended daily calcium intake for children 4-8 years old is 800 mg. One glass of milk (200 ml) contains approximately 296 mg of calcium. Which percent of the recommended calcium daily intake contains one glass of milk?

2. Find the minimum integer value of x for which the following inequality holds:

$$\frac{1}{x+3} - \frac{1}{x+4} \geq \frac{1}{x+6} - \frac{1}{x+7}$$

3. Solve the inequality: $(x-1)^2 - 10|x-1| + 21 \leq 0$. Write down the sum of all integer solutions.

4. Real numbers x and y satisfy the following inequalities system:

$$\begin{cases} x^2 - y^2 \geq 40 \\ 0 \leq x - y \leq 4 \end{cases}$$

   We call *weight* of a pair of numbers (x, y) the value $x^2 + y^2$. Find all such pairs (x, y) that the expression $x + y$ atteins its minimum possible value. Compute the total weight of all such pairs. *For example if the found pairs are* $(1, 2)$ *and* $(3, 4)$ *then the resulting weight is* $30 = (1^2 + 2^2) + (3^2 + 4^2)$.

5. Find the perimeter of a convex polygon with vertices at (all) solutions of the following equation:

$$((|x| - 12)^2 + (|y| - 12)^2)((|x + y| - 17)^2 + (|x - y| - 17)^2) = 0$$

6. A stick of length 13 has 12 marks (on the same distance 1 from each other and from the ends of the stick). Find the number of ways to break the stick in two of the 12 marks in such a way that a triangle could be constructed from the resulting 3 parts (so that the triangle's inequality holds).

7. A circle is inside a rectangle $ABCD$. Tangent segments AQ, BP, CN, DM are constructed to the circle. It turned out that $BP = 8, CN = 10, DM = 7$. Compute AQ^2.

8. On the first day back to school 30 students were lined up near their class. It turned out that each student with the index in the line from 2 to 15 (inclusive, counting from left to right) has the number of friends in the line with the index more than i greater by 1 than the number of friends with the index less than i.

   For the students with the index from 16 to 29 (inclusive, counting from left to right) a similar property holds: for each student with the index i the number of friends in the line with the index less than i is greater by 2 than the number of friends with the index greater than i.

   It turned out that the first student (with the index 1) has 19 friends. How many friends has the last student (with the index 30)?

9. Find all such integer numbers n that $1 < n < 10^6$ and $n^3 - 1$ is divisible by $10^6 n - 1$.

10. Consider all possible sets containing 49 different positive integer numbers less or equal to 100. Stepan assigned a positive integer tag less or equal to 100 to each of these sets (different sets can have equal tags). Prove that such set L exists which contains 50 different positive integer numbers less or equal to 100 that for any number x in L the tag assigned to the set $L - \{x\}$ is not equal to x.

   Here $L - \{x\}$ denotes the set L excluding the element x.