HSE Global Scholarship Competition – 2021
Mathematics
11th grade, variant 2

1. In year 2019 and in year 2020 Pavel passed the same pack of 6 examinations (one exam in each of 6 different subjects). He noticed that on the second year his marks repeated in 5 of the six subjects (still could differ from a subject to subject). In one more subject his result in 2019 was 86 points and it was 68 points in 2020. In 2019 his mean average for the six subjects was equal to 84. What is his mean average in 2020?

2. We denote by x_0 the maximum value of x for which the inequality
$$\frac{x^2+4x+1}{x^2-4x+1} > 0$$
does **not** hold. Compute $x_0^2 + \frac{1}{x_0^2}$.

3. Find the minimum value of the expression $x - \sqrt{x + \frac{19}{4}}$.

4. Compute $\left\lfloor \frac{3^0}{5} \right\rfloor + \left\lfloor \frac{3^1}{5} \right\rfloor + \left\lfloor \frac{3^2}{5} \right\rfloor + \ldots + \left\lfloor \frac{3^{11}}{5} \right\rfloor$. Give the answer in a form of an integer number.

   By $\lfloor x \rfloor$ we denote the function of (lower) integer part of a number x, i.e., by definition, it is the maximum integer number not exceeding x.

5. Consider a geometric progression $b_n = a^n$. Find the number of such integer numbers a that for the sequence $\{b_n\}$ defined by this a the inequality holds:
$$\frac{1}{b_3} - \frac{6}{b_2} + \frac{6}{b_1} - \frac{1}{b_0} \geq 0$$

6. A number is chosen at random from 1 to 1023. We denote α the probability that in binary notation the number has 3 unit digits. Compute $\frac{10}{\alpha}$.

7. Ten numbers, not necessarily different, are written on a blackboard. The following 10 sums were computed: sum of all the 10 numbers except for the first one, sum of all the 10 numbers except for the second one, and so on, until the last sum: sum of all the 10 numbers except for the 10th number. As the results 9 different values occured: $86, 87, 88, 89, 90, 91, 92, 93, 96$. Find the initial ten numbers. Write down the 4th number in the increasing order of them as the answer.

8. Let angles A and B of a triangle ABC be equal $90°$ and $75°$ respectively and length of side AB equal to 3. Points P and Q are picked up on the sides AC and BC respectively such that $\angle APB = \angle CPQ$ and $\angle BQA = \angle CQP$. Compute QA^2 (i.e. the square of the length of a segment QA).

9. Three players Feodor (F), Sergey (S) and Tanya (T) take turns in a game by taking stones from a pile with n stones. They take turns one after another in the order F, S, T, F, S, T, ... Feodor starts the game and the person who takes the last stone loses. Feodor and Tanya have united against Sergey and try to work out a joint strategy. Sergey can take any quantity

of stones from 1 to 7 in his turn while Feodor and Tanya can take (each of them) 1, 2, 3 or 4 stones in their turn. Determine for which values of n there exists a winning strategy for the team of Feodor and Tanya (i.e. when Sergey loses) and for which values the winning strategy exists for Sergey.

10. Prove that for an even positive integer number $n \geq 10$ there exists exactly 4 such positive integer numbers k that $n + k^2$ is divisible by $n + k$ if and only if $n = 2p$ where p and $2p + 1$ are prime numbers.