

HSE Global Scholarship Competition – 2020

Mathematics

11th grade, variant 3

1. Points P and Q are the two opposite vertices of a cube with an edge length 6. Two balls of radii 1 and 2 are inside of the cube. One of them touches all the three faces of the cube containing P and another touches all the three faces containing Q . Find the distance between the balls' centers.
2. Find the total of intervals of negative numbers which satisfy the following condition:
$$\frac{2\sqrt{x+3}}{x+1} \leq \frac{3\sqrt{x+3}}{x+2}.$$
3. Find the maximal remainder of division a square trinomial $-x^2 - x + 13$ by a linear binomial $4x - a$ (among all real values of the parameter a).
4. For an arbitrary real $x \neq 0$ and a function $f(x)$ the following condition holds: $f\left(\frac{x^2+49}{x}\right) = \frac{4x-2x^2-98}{x^2+49}$. Compute $f(8)$.
5. Compute the sum $[\sqrt{1}] + [\sqrt{2}] + [\sqrt{3}] + \dots + [\sqrt{48}] + [\sqrt{49}]$.
By brackets we denote the lower integer part of a real number
6. How many different positive integer divisors of a number $2^5 \cdot 3^3 \cdot 5^2$ have themselves an odd number of positive integer divisors?
7. Let point O be a center of a circle with radius equal to 5. AB is a chord with length 6 in the circle. A square $PQRS$ is inscribed to a sector AOB so that point P belongs to a segment OA , point Q belongs to a segment OB and points R and S belong to the circle. Find the area of the square $PQRS$.
8. A word in language Ababab is any sequence of letters 'a' and 'b'. Two words in the language are considered *similar* if one of them can be derived from another by repeating the following operation: we choose several consecutive letters containing even number of 'a'-s from a word and put them to the same place in the word but in the opposite order (for example, "aba**ab**ba" \rightarrow "ab**ba**aba"). How many dissimilar words of length 10 does there exist in Ababab language?
In other words: what is the maximal number of pairwise not similar words of length 10.
9. Find all positive integers n so that each integer written down in decimal record with $n - 1$ digits '1' and one digit '7' is a prime number (for example 1711 is not because $1711 = 29 \cdot 59$).

10. A map of the Central swamp looks like following: there is a stone in the center of the swamp which is denoted with point O on the map where the Queen Frog sits. Two concentric circles Γ_1 (closer to O) and Γ_2 (further from O) surround O . A 100 radii with common endpoint O intersect these circles. Firstly they intersect Γ_1 in points B_1, B_2, \dots, B_{100} and then they intersect Γ_2 in points D_1, D_2, \dots, D_{100} respectively. The points are mentioned as they follow along the circles and points B_i and D_i (with same indices) belong to the same radius.

The points B_i and D_j denote 200 bumps of the swamp on the map. Totally 401 frogs are sitting on the bumps. If at least 3 frogs sit on the same bump then 3 of them will jump simultaneously to the neighbouring bumps (each frog to a different bump) some time later. Prove that at some point the following propertie will hold: for each bump either it has a frog on it or its neighbouring bump has a frog on it.

By neighbouring bumps we call, at first, bumps with the same letter and neighbouring indices (1 and 100 are also neighbouring), at second, bumps with different letters but the same indices.