Reference Materials
Mathematics

10th-11th Grades

I. Methodological recommendations for preparing for the competition

Competitions with nonstandard problems are important tradition of international mathematical education. Many taskbooks with competition problems of different difficulty are published nowadays. Anyone can get acquainted with problems of such fashion, test himself in solving the problems personally and train from those books. Huge experience in dealing with nonstandard mathematical high school level problems has led to creation of an informal “competition minimum” programme. That is a set of tricks, methods and theorems which proved to be useful on a competition. Oftentimes those are out of high school programme but still could be proved with just high school methods.

The main source of such “competition minimum” are textbooks either with competition problems grouped into collections in the same topic or with materials of various school mathematical circles (optional classes). Please, take into account that oftentimes such textbooks show adapted to school level and polished solutions of very hard problems. Even though such reading is extremely useful for developing mathematical skills, mastering this material to this extent is not necessary for solving competition problems.

The HSE Global Scholarship Competition (GSC) in Mathematics is held both in in-person format (offline) and remote format (online). Each participant may choose just one of the formats: writing either on a special venue or remotely. Either way a participant has 180 minutes for writing the test.

A task of GSC-2022 in Mathematics consists of **10 problems** divided into three blocks according to the problems’ difficulty:

- Problems of the *first block* (from 1st to 6th) are graded **7 points** for correct answer and 0 **points** for incorrect one. Just **answers** are graded in this block.
- Problems of the *second block* (7th and 8th) are marked in a range **from 0 to 13 points**. Both **answer and scheme of a proof** are graded in this block.
- Problems of the *third block* (9th and 10th) are marked in a range **from 0 to 16 points**. **Full solutions** are supposed in this block. Their grades account for correctness, fullness, and integrity of the solutions.

Thus, problems number 7 to 10 suppose writing a partial or full solution. Online participants enter their solutions to a special web form on the competition platform using a standard keyboard. Scheme of a proof mentioned in the second block supposes listing all the main steps of the solution without mentioning excessive details. Please find an example of such scheme further in this file (**).

Task difficulty is mostly a subjective concept, it depends on interests and skill set of each competitor. Nevertheless, speaking about averaged difficulty, the difficulty of problems grows from the first problem to the last one. Problems of *the first block* are closer to school examinations. They test competitor's skills in concepts and methods of high school programme. Problems of *the second block* are out of scope of a standard high school programme but they could be found on classes of advanced math as more technical tasks. Problems of *the third block* are of a high competition level comparable to the finals of national mathematical competitions and of the International mathematical olympiad (IMO).

For solving most of the competition tasks it is enough to know school high programme which we mention further. However, competition problems oftentimes need not just knowledge but also creativity, reasoning skills, analyzing the computational results, conjecturing statements and verifying the conjectures.

II. Key notions of high school programme in Mathematics for preparations to the competition (as a reference for *the first block*: problems 1 – 6)

Grade 10:

Numbers: from integers to reals, notations, axiomatic approach, divisibility of integer numbers, GCF and LCM, prime decomposition.

Functions: linear, rational, defined by conditions, exponential.

Function's properties: graph of a function, shifts of a graph, increasing/decreasing, odd/even functions, minimum/maximum, periodic functions.

Inequalities: linear and square, rational inequalities and the method of intervals, using properties of functions for comparing values.

Equalities: linear, square, Vieta theorem, rational equalities, compound interest (compound percentage), systems of equations.

Sequences: arithmetic and geometric progressions, sum of elements of a progression.

Combinatorics and probability theory: exhaustive search, the complementary set, inclusion-exclusion principle, choose numbers/permutations/arrangements, mathematical expectation, combinatorial definition of probability, geometric probability.

Geometry of a triangle: congruence and similarity conditions, auxiliary objects inside a triangle: median, height, angle bisector, incircle and circumcircle, 'wonderful' triangle points.

Arithmetic of angles: angles and parallel lines, inscribed and central angles of a circle, angle between tangent and chord, specific marks on drawings for counting angles.

Computations in a triangle: triangle inequality, sin and cos theorems, Ceva and Menelaus theorems, expressions for basic elements of a triangle using sides and angles, area of a triangle and of more complex figures.

Vector-coordinate approach: basic vector operations, coordinate definitions for basic geometric objects, scalar product.

Geometric transformations: plane movements, movements composition, homothety.

Grade 11*:**

Numbers: from integers to reals, notations, axiomatic approach, divisibility of integer numbers, GCF and LCM, prime decomposition.

Functions: linear, rational, defined by conditions, exponential, *trigonometric, polynomials and n-th roots, inverse function, division of polynomials with a remainder*.

Function's properties: graph of a function, shifts of a graph, increasing/decreasing, odd/even functions, minimum/maximum, periodic functions.

Inequalities: linear and square, rational inequalities and the method of intervals, using properties of functions for comparing values.

Equalities: linear, square, Vieta theorem, rational equalities, compound interest (compound percentage), systems of equations, *trigonometric, roots of polynomials*.

Sequences: arithmetic and geometric progressions, sum of elements of a progression, *sequence limit*.

Derivative: function limit, differentiation rules, using derivative for analysing a function, extremum points, graph tangent equation.

Complex numbers: arithmetic operations, geometric interpretation, exponentiation and taking a root.

Combinatorics and probability theory: exhaustive search, the complementary set, inclusion-exclusion principle, choose numbers/permutations/arrangements, mathematical expectation, combinatorial definition of probability, geometric probability.

Geometry of a triangle: congruence and similarity conditions, auxiliary objects inside a triangle: median, height, angle bisector, incircle and circumcircle, 'wonderful' triangle points.

Arithmetic of angles: angles and parallel lines, inscribed and central angles of a circle, angle between tangent and chord, specific marks on drawings for counting angles.

Computations in a triangle: triangle inequality, sin and cos theorems, Ceva and Menelaus theorems, expressions for basic elements of a triangle using sides and angles, area of a triangle and of more complex figures.

Vector-coordinate approach: basic vector operations, coordinate definitions for basic geometric objects, scalar product.

Geometric transformations: plane movements, movements composition, homothety.

Stereometry: sections, basic bodies and their volume, regular polyhedra, 3d vectors and coordinate approach, parallelism and perpendicularity in 3d space.

III. Recommended sources

Primary sources

1. Fomin D., Genkin S., Itenberg I. Mathematical Circles (Russian Experience). AMS, 1996.

2. Vilenkin N.Ya. Combinatorics. Academic Press, New York, 1971.

3. Prasolov V. Problems in plane and solid geometry. Plane Geometry.

Additional sources

1. Fedorov R., Belov A., Kovaldzhi A., Yashchenko I. Moscow Mathematical Olympiads, 1993-1999. AMS, 2011.
2. Fedorov R., Belov A., Kovaldzhi A., Yashchenko I. Moscow Mathematical Olympiads, 2000-2005. AMS, 2011.
3. Taylor P. J. International Mathematics Tournament of the Towns,
   Book 1: 1980-1984. Australian Mathematics Trust, 1993.
   Book 2: 1984-1989. Australian Mathematics Trust, 2003.
   Book 3: 1989-1993. Australian Mathematics Trust, 1994.
4. Taylor P. J., Storozhev A. M. International Mathematics Tournament of Towns,
   Book 4: 1993-1997. Australian Mathematics Trust, 2003.
5. Storozhev A. M. International Mathematics Tournament of Towns,
   Book 5: 1997-2002. Australian Mathematics Trust, 2006.
6. Andreescu T., Feng Z. 102 Combinatorial Problems. Birkhauser, 2002.
7. Andreescu T., Andrica D. Number Theory: Structures, Examples, and Problems. Birkhauser, 2009.
8. Andreescu T., Andrica D., Feng Z. 104 Number Theory Problems. Birkhauser, 2007.
9. Andreescu T., Enescu B. Mathematical Olympiad Treasures. Birkhauser, 2012.
10. Coxeter H. S. M., Greitzer S. L. Geometry Revisited. AMS, 1967.
11. Engel A. Problem-Solving Strategies. Springer, 1999.

IV. Online sources

International Mathematical Olympiad (official web site). URL: www.imo-official.org (‘Problems’ section: tasks and short lists for the previous Olympiads)

*****An example of a scheme of a proof for tasks of the second block:***

Problem 7 (demonstration task HSE GSC-2016):

Find all such pairs of relatively prime integer positive a and b, that a number $2a^2 + 3b^2$ is divisible by $2a + 3b$.

Proof scheme:

1. Consider differences $2a^2 + 3b^2 - a(2a + 3b)$ and $2a^2 + 3b^2 - b(2a + 3b)$, each divisible by $2a + 3b$. Conclude $(b - a) \cdot GCF(2a, 3b)$ is divisible by $2a + 3b$.
2. Convert to the form: $(b - a) \cdot GCF(2, b) \cdot GCF(3, a)$ is divisible by $2a + 3b$.
3. If $a = b$ then as they are relatively prime, we deduce $a = b = 1$.
4. If $GCF(2, b) \cdot GCF(3, a) \leq 2$ then we get a contradiction using the following property: if a positive integer n is divisible by m then $n \geq m$.
5. If $GCF(2, b) \cdot GCF(3, a) = 3$ then by subtracting $2a + 3b$ from $(b - a) \cdot GCF(2, b) \cdot GCF(3, a)$ we get $5a$ is divisible by $2a + 3b$.
    - By considering such a' that $a = 3a'$ we get one valid solution (6, 1) and one extraneous solution.
6. If $GCF(2, b) \cdot GCF(3, a) = 6$ then similarly we get $10a$ is divisible by $2a + 3b$.
    - Using substitution $b = 2b'$ and $a = 3a'$ we get 5 is divisible by $a' + b'$. Direct check gives two extraneous solutions and two proper solutions: (3, 8) and (9, 4).

Answer: the solutions are (1,1); (6,1); (3,8); (9,4).

********Italic font*** is used for topics which were added to the programme of grade 11 on top of the programme of Grade 10.**